Student Number ______________________________ Teacher ______________________

NORTH SYDNEY BOYS
HIGH SCHOOL

2022	NSBHS TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks : 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8 – 34)

- Attempt Questions 11 – 33
- Allow about 2 hours and 45 minutes for this section.

(For marker's use only)

SECTION	SECTION I (MC)	SECTION II (Part 1)	SECTION II (Part 2)	TOTAL
MARK	$\overline{10}$	$\overline{38}$	$\overline{52}$	$\overline{100}$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1. $2\log_{10}(x) - \log_{10}(3x)$ is equal to

A. $\log_{10}\left(\frac{x}{3}\right)$

B. $\log_{10}(x^2 - 3x)$

C. $\frac{2\log_{10}(x)}{\log_{10}(3x)}$

D. $-\log_{10}(x)$

2. The second derivative of the function $f(x)$ is given by $f''(x) = \frac{2x}{1+x^2}$

The interval on which the graph of $f(x)$ is concave up is

A. $x < 0$

B. $x \leq 0$

C. $x > 0$

D. $x \geq 0$

3. How many terms are there in the following geometric sequence?

$$3, 6, 12, 24, \dots, 384$$

A. 7

B. 8

C. 9

D. 10

4. The solution to the inequality $6 - x - x^2 \leq 0$ is

A. $x \leq -3$ or $x \geq 2$

B. $x \leq -2$ or $x \geq 3$

C. $-3 \leq x \leq 2$

D. $-2 \leq x \leq 3$

5. If $P(A) = 0.8$, $P(B) = 0.5$ and $P(B|A) = 0.4$,

What is the value of $P(A \cap B)$?

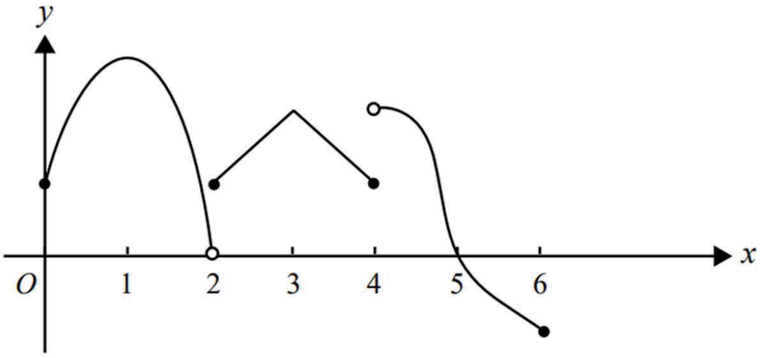
A. 0.32

B. 0.25

C. 0.1

D. 0.5

6. The graph of the function $f(x)$ with domain $x \epsilon [0, 6]$ is shown below.



Which of the following is **not** true?

A. The function is not continuous at $x = 2$ and $x = 4$.

B. The function exists for all values of x between 0 and 6.

C. $f(x) = 0$ for $x = 2$ and $x = 5$

D. The function is positive for $x \epsilon [0, 5)$

7. Determine $\int \frac{e^x + 1}{e^x}\, dx$

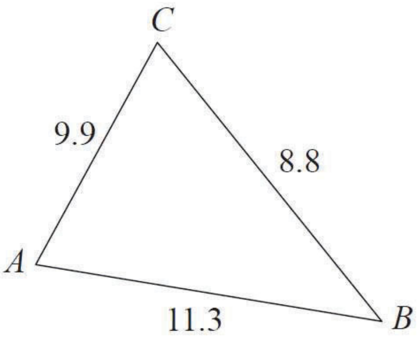
A. $x - e^{-x} + C$

B. $x + e^{-x} + C$

C. $1 + xe^{-x} + C$

D. $x + xe^{-x} + C$

8. Determine the size of angle A in the following triangle.



Not drawn to scale

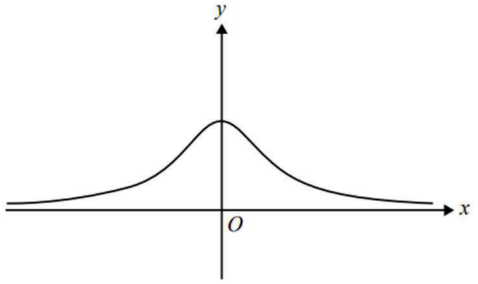
A. $48.5°$

B. $61.4°$

C. $118.6°$

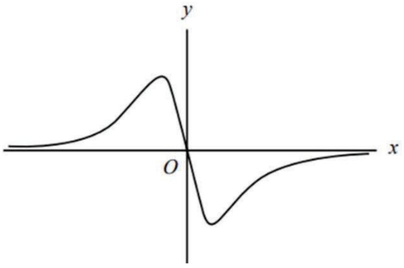
D. $131.5°$

9. The graph of a function $f(x)$ is shown below

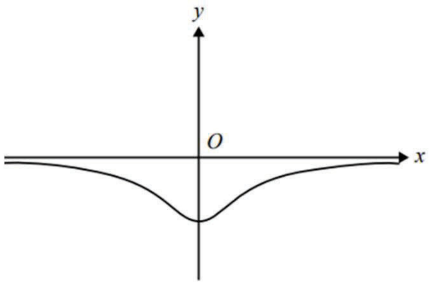


The graph of the **antiderivative** of $f(x)$ could be:

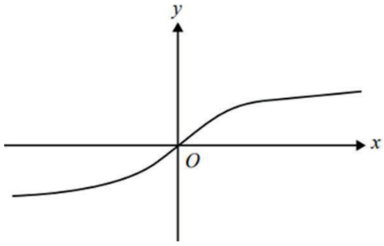
A.



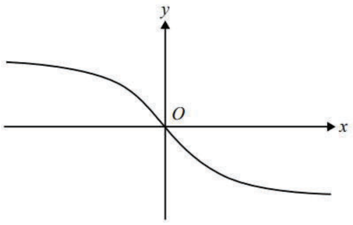
C.



B.



D.



10. If A and B are two **independent events**, then the probability of occurrence of at least one of A and B is given by:

A. $1 + P(\bar{A})P(\bar{B})$

B. $1 - P(\bar{A})P(\bar{B})$

C. $P(A) + P(B) - P(\bar{A})P(\bar{B})$

D. $1 - P(A)P(B)$

Question 11 (1 mark)

Write 132° in radian measure **1**

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Question 12 (3 marks)

A discrete random variable has the following probability distribution. **3**

x	1	2	3	4
$P(X = x)$	0.5	0.2	0.1	0.2

Find the **expected value** and **variance** of X

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Question 13 (6 marks)

Differentiate the following with respect to *x*:

(a) $y = \sqrt{4 - x^2}$ **2**

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(b) $y = \dfrac{\log_e x}{6x}$ **2**

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(c) $y = e^{2x} \sin x$ **2**

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Question 14 (6 marks)

(a) Find $\int 6x + 7\cos\left(\frac{x}{2}\right)\, dx$ **2**

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(b) Find $\int \frac{x + 2}{x^2 + 4x - 6}\, dx$ **2**

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(c) Evaluate $\int_0^2 3^{-x}\, dx$ **2**

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Question 15 (3 marks) **3**

Solve $\sqrt{3}\sin x = -\cos x$ for $x \in [0\,,3\pi]$

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Question 16 (3 marks) **3**

If $\sin\theta = -\dfrac{12}{13}$, find the values of $\cos\theta$ and $\tan\theta$ where $\theta \in \left[\pi, \dfrac{3\pi}{2}\right]$

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Question 17 (2 marks)

Show that $\cot A + \dfrac{\sin A}{1 + \cos A} = \operatorname{cosec} A$ **2**

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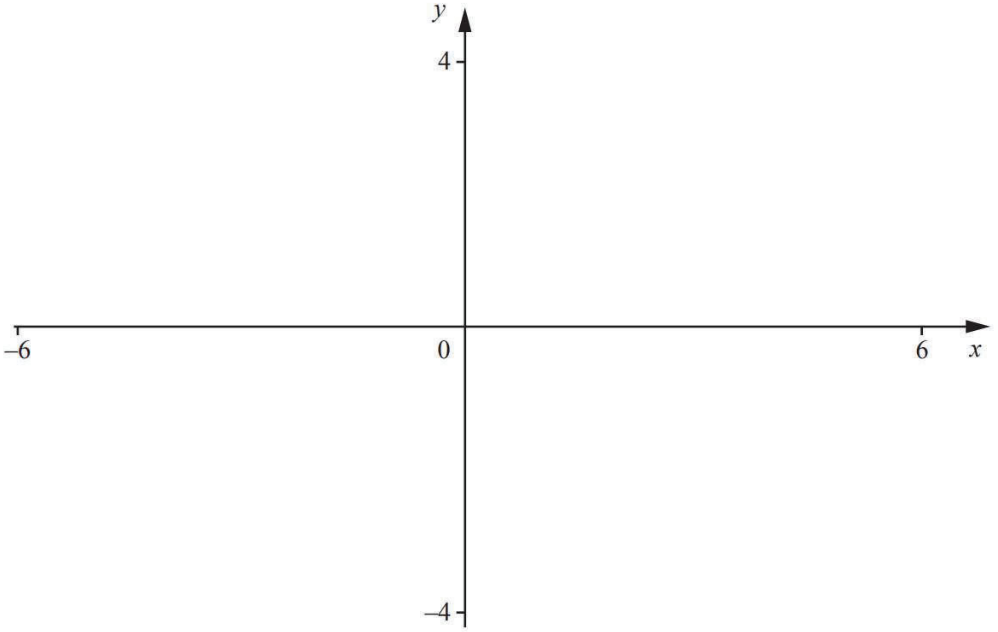
Question 18 (5 marks)

(a) On the axes below, sketch the graphs of $y = |x - 3|$ and $y = \left|\frac{2}{5}x\right|$, giving the coordinates of the points where the graphs meet. **4**



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(b) Hence, solve the inequation $\left|\frac{2}{5}x\right| < |x - 3|$ **1**

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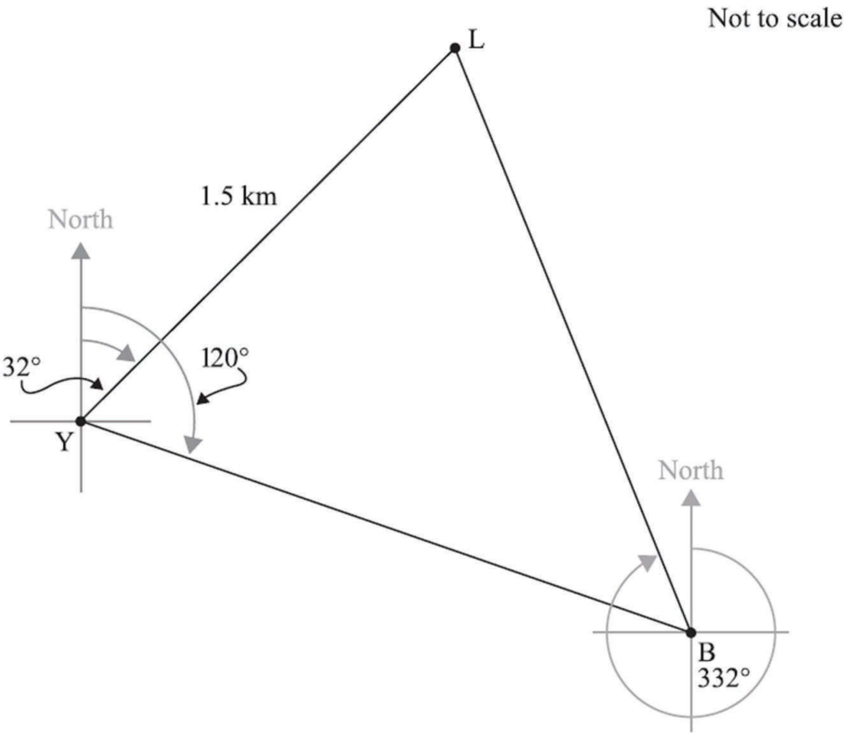
Question 19 (6 marks)

A yacht is located at point Y and is sailing on a bearing of $032°T$ towards a lighthouse at point L 1.5 km from point Y. From Y, the yacht's navigator spots a boat at point B bearing $120°T$. The bearing of the lighthouse from the boat is $332°T$.



(a) Calculate the distance between the yacht and the boat. **3**

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(b) From the boat, the top of the lighthouse has an elevation of 2°. Determine the height of the lighthouse above sea level. **3**

Question 20 (3 marks)

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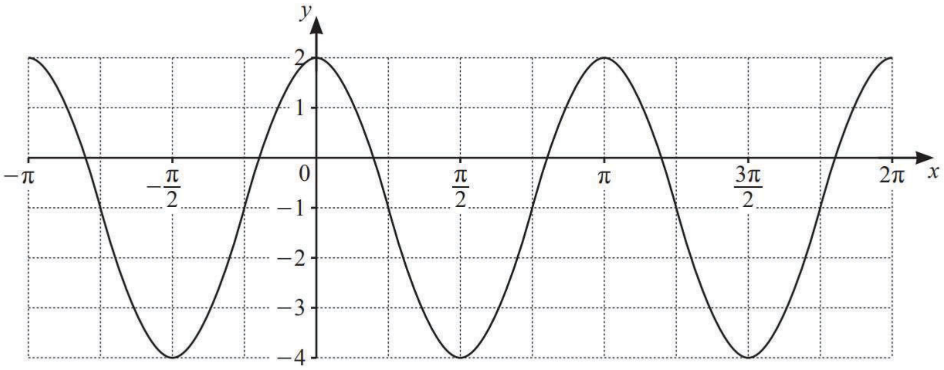

The curve has equation $y = a\cos bx + c$ where a, b, c are integers. Find the values of a, b and c.

End of Questions in Answer Booklet 1

Question 21 (3 marks)

Solve the following equation for x **3**

$$5^x - \frac{8}{5^x} = 2$$

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Question 22 (3 marks) **3**

Use the trapezoidal rule with 4 intervals to evaluate $\int_{0.5}^{2.5} \sqrt[3]{\log_e x}\, dx$ correct to 3 decimal places

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Question 23 (5 marks)

(a) Find the x coordinate(s) of the stationary point(s) on the curve $y = 3\log_e x + x^2 - 7x$, where $x > 0$. **2**

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(b) Hence, determine the nature of each of the stationary point(s) **3**

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Question 24 (4 marks)

There are some red counters and some white counters in a bag. At the start, 7 of the counters are red and the rest of the counters are white. Woody takes two counters from the bag. First he takes at random a counter from the bag. He does not put the counter back in the bag. Woody then takes at random another counter from the bag.

(a) Let the number of white counters in the bag be x. **2**

Draw a tree diagram that represents the above scenario, showing all relevant information.

(b) It is given that the probability that the first counter Woody takes is white, **and** the second counter Woody takes is red is $\frac{21}{80}$. **2**

Find the number of white counters in the bag at the start.

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Question 25 (6 marks)

A geometric sequence is such that its sum of the first 4 terms is 17 times its sum of the first 2 terms. It is given that the common ratio of this geometric sequence is positive and not equal to 1.

(a) Find the common ratio of this geometric sequence. **3**

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(b) Given that the 6th term of this geometric sequence is 64. Find the first term. **2**

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(c) Does this sequence have a limiting sum? Explain your reasoning. **1**

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Question 26 (3 marks) **3**

Determine the number of solutions for $\sin\theta = -0.7$, where $\theta \in [0, 51\pi]$ and provide reasons for your answer.

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Question 27 (3 marks) **3**

The Richter scale defines the magnitude of an earthquake as $M = \log_{10}\left(\frac{I}{S}\right)$ where I is the intensity of the earthquake wave, and S is the intensity of the smallest detectable wave.

An earthquake that registered 6.4 in magnitude was followed by another which was 4 times more intense.

Determine the magnitude of the second earthquake accurate to 1 decimal place.

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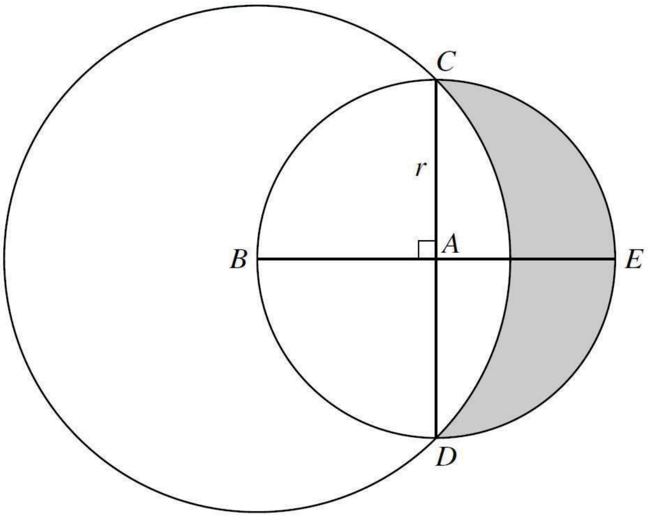
Question 28 (3 marks) **3**



The above diagram shows a circle with centre A and radius r. Diameters CAD and BAE are perpendicular to each other. A larger circle has centre B and passes through C and D.

Find the area of the shaded region in terms of r.

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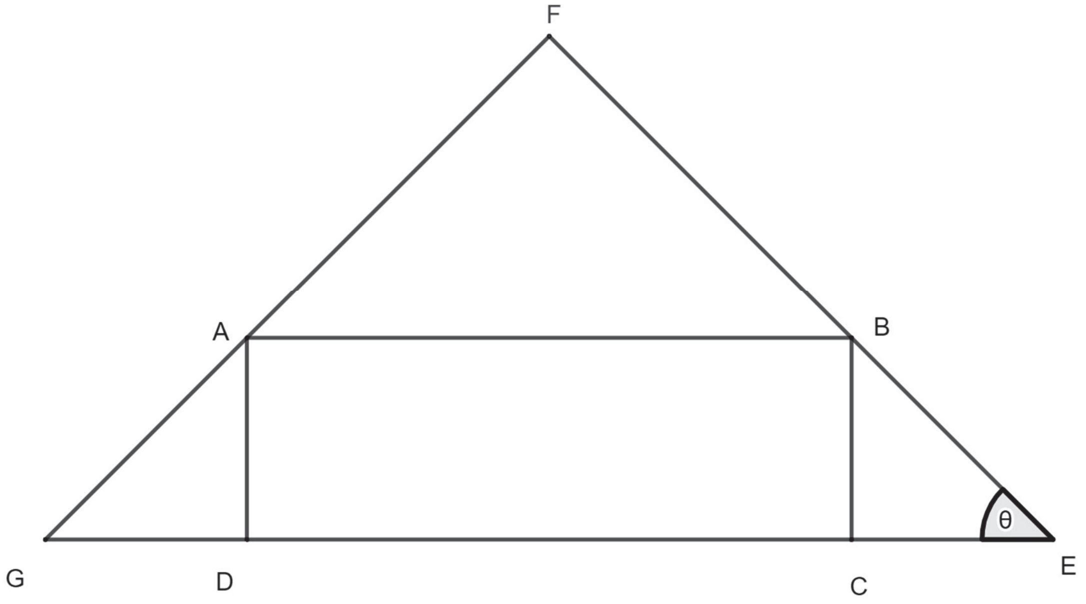
Question 29 (7 marks)



A rectangle $ABCD$ of length 6cm and width 2cm is inscribed in an isosceles triangle EFG, where $FG = FE.$

Let $\angle FEG = \theta$

(a) Show that the area of the isosceles triangle can be expressed as **3**

$$A = 12 + 9\tan\theta + 4\cot\theta$$

(b) Hence, find the minimum area of the isosceles triangle that inscribes the rectangle. **4**

Question 30 (4 marks) **4**

The first four numbers of an arithmetic sequence are $p, 9\ , 3p - q, 3p + q$.

Find the 2022th term.

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Question 31 (6 marks)

(a) Given that $y = xe^{-x}$, find $\frac{dy}{dx}$ and hence show that **3**

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

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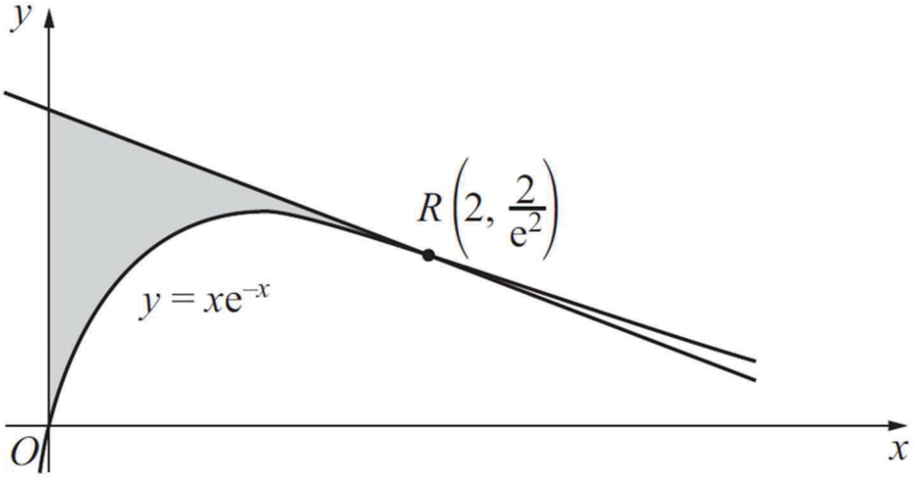
(b) **3**



The diagram above shows part of the curve $y = xe^{-x}$ and the tangent to the curve at the point $R\left(2, \frac{2}{e^2}\right)$.

Find the area of the shaded region bounded by the curve, the tangent and the y axis. Leaving your answer in exact form.

Question 32 (2marks)

Given that $E(aX+b)=aE(X)+b$, where $E(X)$ is the expected value of a discrete random variable X and a and b are constants. **2**

Prove that $Var(aX+b)=a^2\ Var(X)$

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Question 33 (3 marks) **3**

At the point $(2,-5)$ on the curve $y=f(x)$, the tangent has the equation $2x-y-9=0$. Determine the equation of the tangent to the curve $y=4-2f\left(3+\frac{x}{2}\right)$ at the point $(-2,14)$, showing all reasoning.

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End of Paper

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1. $2\log_{10}(x) - \log_{10}(3x)$ is equal to

A. $\log_{10}\left(\frac{x}{3}\right)$

B. $\log_{10}(x^2 - 3x)$

C. $\frac{2\log_{10}(x)}{\log_{10}(3x)}$

D. $-\log_{10}(x)$

$$\log_{10}(x^2) - \log_{10}(3x)$$
$$= \log_{10}\left(\frac{x^2}{3x}\right)$$
$$= \log_{10}\left(\frac{x}{3}\right)$$

Ⓐ

2. The second derivative of the function $f(x)$ is given by $f''(x) = \frac{2x}{1+x^2}$

The interval on which the graph of $f(x)$ is concave up is

A. $x < 0$

B. $x \leq 0$

C. $x > 0$

D. $x \geq 0$

concave up
when $f''(x) > 0$

$\therefore \; 2x > 0$
$x > 0$

Ⓒ

3. How many terms are there in the following geometric sequence?

$$3, 6, 12, 24, \ldots, 384$$

A. 7

B. 8

C. 9

D. 10

$T_n = ar^{n-1}$
where $a = 3$ $r = 2$
$384 = 3 \times 2^{n-1}$
$128 = 2^{n-1}$
$2^7 = 2^{n-1}$
$\therefore n = 8$

Ⓑ

4. The solution to the inequality $6 - x - x^2 \leq 0$ is

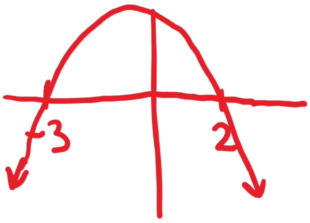
A. $x \leq -3$ or $x \geq 2$

B. $x \leq -2$ or $x \geq 3$

C. $-3 \leq x \leq 2$

D. $-2 \leq x \leq 3$

$(3+x)(2-x) \leq 0$



$\therefore$ $x \leq -3$ or $x \geq 2$

Ⓐ

5. If $P(A) = 0.8$, $P(B) = 0.5$ and $P(B|A) = 0.4$,

What is the value of $P(A \cap B)$?

A. 0.32

B. 0.25

C. 0.1

D. 0.5

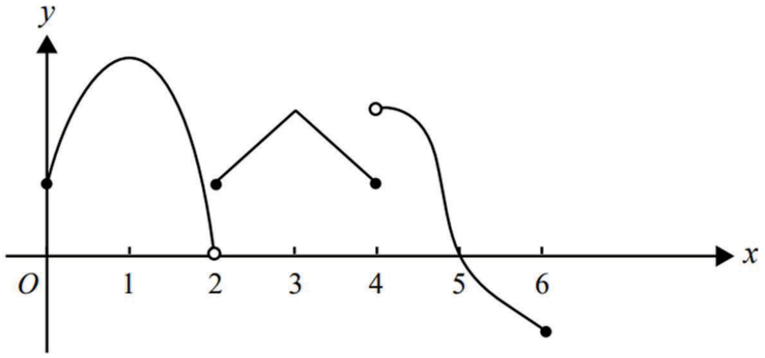
$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$0.4 = \frac{P(B \cap A)}{0.8}$$

$$P(B \cap A) = 0.4 \times 0.8$$
$$= 0.32$$

Ⓐ

6. The graph of the function $f(x)$ with domain $x \in [0, 6]$ is shown below.



Which of the following is **not** true?

A. The function is not continuous at $x = 2$ and $x = 4$. True

B. The function exists for all values of x between 0 and 6. True

C. $f(x) = 0$ for $x = 2$ and $x = 5$ False, at $x=2$ $y>0$

D. The function is positive for $x \in [0, 5)$

Ⓒ

7. Determine $\int \frac{e^x + 1}{e^x}\, dx$

A. $x - e^{-x} + C$

B. $x + e^{-x} + C$

C. $1 + xe^{-x} + C$

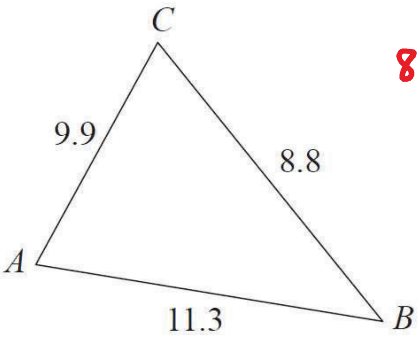
D. $x + xe^{-x} + C$

$$\int 1 + e^{-x}\, dx$$

$$= x - e^{-x} + C$$

Ⓐ

8. Determine the size of angle A in the following triangle.



$8.8^2 = 9.9^2 + 11.3^2 - 2 \times 9.9 \times 11.3 \cos A$

$\cos A = \frac{9.9^2 + 11.3^2 - 8.8^2}{2 \times 9.9 \times 11.3}$

$A \doteqdot 48.5°$

Not drawn to scale

Ⓐ

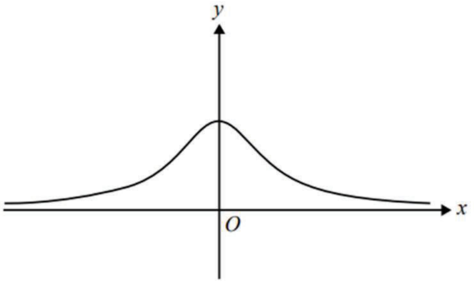
A. 48.5°

B. 61.4°

C. 118.6°

D. 131.5°

9. The graph of a function $f(x)$ is shown below

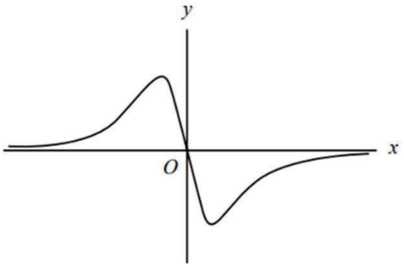


$f'(x) > 0$

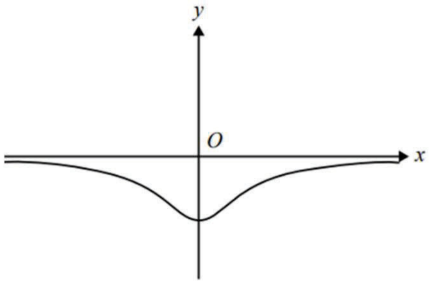
$\therefore f(x)$ is strictly increasing

The graph of the **antiderivative** of $f(x)$ could be:

A.



C.



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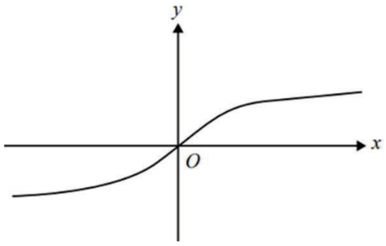
B.



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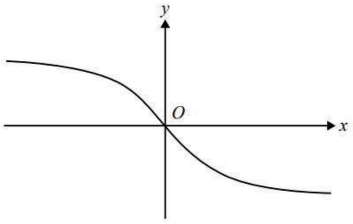

10. If A and B are two **independent events**, then the probability of occurrence of at least one of A and B is given by:

A. $1 + P(\bar{A})P(\bar{B})$

B. $1 - P(\bar{A})P(\bar{B})$

C. $P(A) + P(B) - P(\bar{A})P(\bar{B})$

D. $1 - P(A)P(B)$

at least one of A and B is equivalent to $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B)$$

(For independent events,

$P(A \cap B) = P(A) \cdot P(B)$

$$= P(A) + P(B)(1 - P(A))$$

$$= P(A) + P(B) \cdot P(\bar{A})$$

$$= 1 - P(\bar{A}) + P(B) \cdot P(\bar{A})$$

$$= 1 - P(\bar{A})[1 - P(B)]$$

$$= 1 - P(\bar{A})P(\bar{B})$$

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Question 11 (1 mark)

Write 132° in radian measure **1**

$132^\circ = \frac{132}{180}\pi$

$= \frac{11}{15}\pi$

Question 12 (3 marks)

A discrete random variable has the following probability distribution. **3**

x	1	2	3	4
$P(X = x)$	0.5	0.2	0.1	0.2

Find the **expected value** and **variance** of X

x	1	2	3	4	Sum
$xP(X=x)$	0.5	0.4	0.3	0.8	2
$x^2P(X=x)$	0.5	0.8	0.9	3.2	5.4

$E(x) = \sum x P(X=x)$

$= 2$

$Var(x) = E(X^2) - [E(x)]^2$

$= 5.4 - 2^2$

$= 1.4$

Question 13 (6 marks)

Differentiate the following with respect to x:

(a) $y = \sqrt{4 - x^2}$ **2**

$y = (4 - x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}} \cdot -2x$

$= -\frac{x}{\sqrt{4 - x^2}}$

(b) $y = \frac{\log_e x}{6x}$ **2**

$\frac{dy}{dx} = \frac{6x \cdot \frac{1}{x} - \log_e x \cdot 6}{(6x)^2}$

$= \frac{6 - 6\log_e x}{36x^2}$

$= \frac{1 - \log_e x}{6x^2}$

(c) $y = e^{2x} \sin x$ **2**

$\frac{dy}{dx} = e^{2x} \cdot \cos x + \sin x \cdot 2e^{2x}$

$= e^{2x}(\cos x + 2\sin x)$

Question 14 (6 marks)

(a) Find $\int 6x + 7\cos\left(\frac{x}{2}\right)\, dx$ **2**

$= 3x^2 + 14\sin\left(\frac{x}{2}\right) + C$

(b) Find $\int \frac{x+2}{x^2+4x-6}\, dx$ **2**

$\frac{1}{2}\int \frac{2x+4}{x^2+4x-6}\, dx$

$= \frac{1}{2}\log_e\left|x^2+4x-6\right| + C$

(c) Find $\int_0^2 3^{-x}\, dx$ **2**

$= -\int_0^2 -3^{-x}\, dx$

$= \left[\ln\left(\frac{1}{3}\right)\cdot\left(\frac{1}{3}\right)^x\right]_0^2$

$= \frac{1}{\log_e 3}\cdot\left(3^{-2} - 1\right)$

$= \frac{1}{\log_e 3}\cdot\left(1 - \frac{1}{9}\right)$

$= \frac{8}{9\log_e 3}$

Question 15 (3 marks) **3**

Solve $\sqrt{3}\sin x = -\cos x$ for $x \in [0, 3\pi]$

$$\frac{\sin x}{\cos x} = -\frac{1}{\sqrt{3}}$$

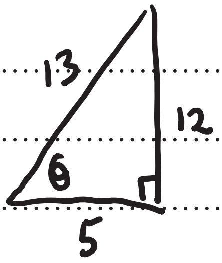
$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = \pi - \frac{\pi}{6},\ 2\pi - \frac{\pi}{6},\ 3\pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6},\ \frac{11\pi}{6},\ \frac{17\pi}{6}$$

Question 16 (3 marks) **3**

If $\sin(\theta) = -\frac{12}{13}$, find the values of $\underline{\cos(\theta)}$ and $\underline{\tan(\theta)}$ where $\theta \in \left[\pi, \frac{3\pi}{2}\right]$



θ is in qudrant 3

$\therefore \cos\theta < 0$ and $\tan\theta > 0$

$$\cos\theta = -\frac{5}{13}$$

$$\tan\theta = \frac{12}{5}$$

Question 17 (2 marks)

Show that $\cot A + \dfrac{\sin A}{1 + \cos A} = \operatorname{cosec} A$ **2**

$$\text{LHS} = \cot A + \frac{\sin A}{1+\cos A}$$

$$= \frac{\cos A}{\sin A} + \frac{\sin A}{1+\cos A}$$

$$= \frac{\cos A + \cos^2 A + \sin^2 A}{\sin A(1+\cos A)}$$

$$= \frac{\cancel{(\cos A+1)}}{\sin A\,\cancel{(1+\cos A)}}$$

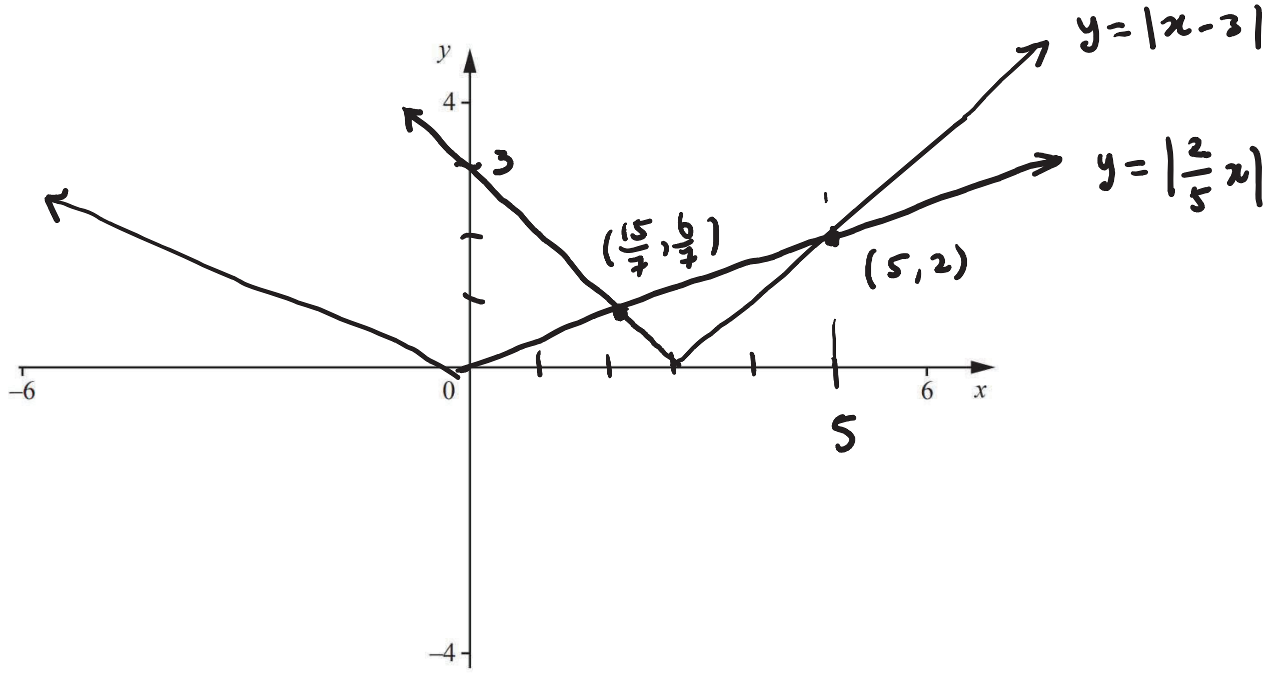
$$= \frac{1}{\sin A}$$

$$= \operatorname{cosec} A$$

$$= \text{RHS}$$

Question 18 (5 marks)

(a) On the axes below, sketch the graphs of $y = |x - 3|$ and $y = \left|\frac{2}{5}x\right|$, giving the coordinates of the points where the graphs meet. **4**



$x - 3 = \frac{2}{5}x$ $\quad$ $x - 3 = -\frac{2}{5}x$

$\frac{3}{5}x = 3$ $\quad$ $\frac{7x}{5} = 3$

$x = 5$ $\quad$ $7x = 15$

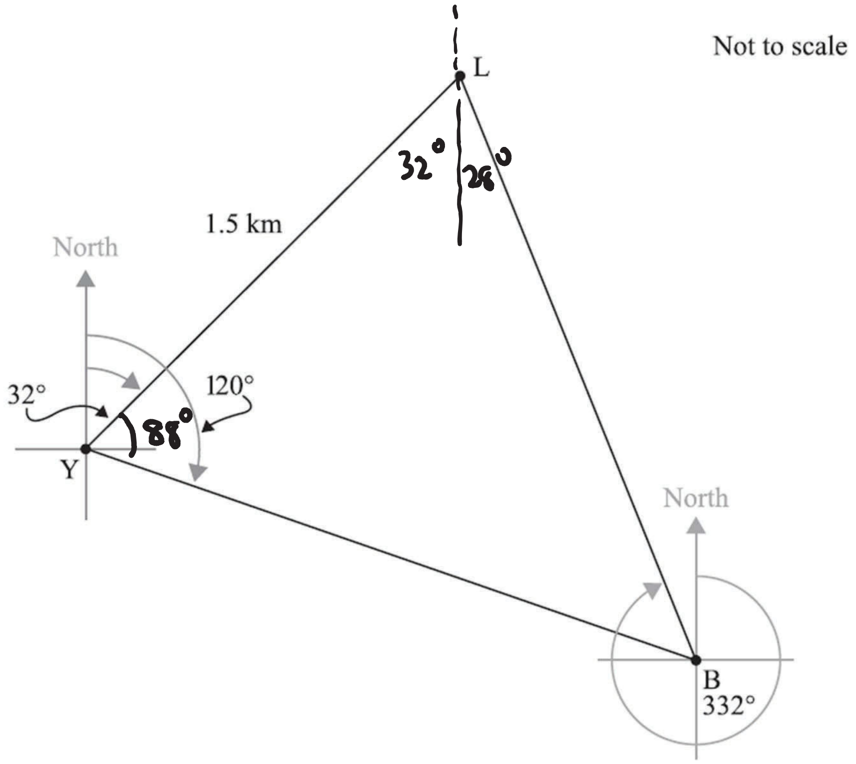
$x = \frac{15}{7}$

(b) Hence, solve the inequation $\left|\frac{2}{5}x\right| < |x - 3|$ **1**

$x < \frac{15}{7}$ OR $x > 5$

Question 19 (6 marks)

A yacht is located at point Y and is sailing on a bearing of $032°T$ towards a lighthouse at point L 1.5 km from point Y. From Y, the yacht's navigator spots a boat at point B bearing $120°T$. The bearing of the lighthouse from the boat is $332°T$.



(a) Calculate the distance between the yacht and the boat. **3**

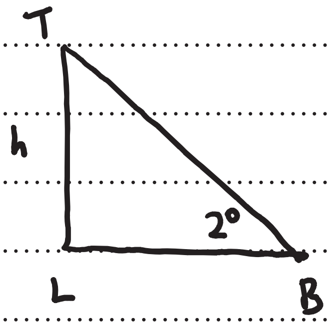
$\angle YLB = 60°$ and $\angle LBY = 32°$

$$\frac{YB}{\sin 60°} = \frac{1.5}{\sin 32°}$$

$$YB = \frac{1.5 \times \sin 60°}{\sin 32°}$$

$$= 2.45 \text{ km}$$

(b) From the boat, the top of the lighthouse has an elevation of 2°. Determine the height of the lighthouse above sea level. **3**



Find LB using sine rule

$\frac{LB}{\sin 88°} = \frac{1.5}{\sin 32°}$

$\therefore LB = \frac{1.5 \sin 88°}{\sin 32°}$

$= 2.83$ km

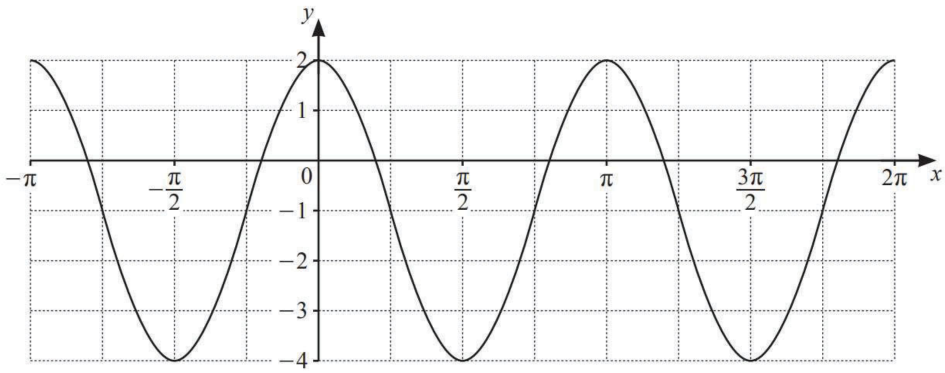
$\therefore \tan 2° = \frac{h}{2.83}$

$\therefore h = 2.83 \times \tan 2°$

$= 99$ m (to nearest m)

Question 20 (3 marks)

3



The curve has equation $y = a\cos bx + c$ where a, b, c are integers. Find the values of a, b and c.

Period:

$T = \frac{2\pi}{n}$

$\therefore \pi = \frac{2\pi}{b}$

$b = 2$

amplitude:

$\frac{2-(-4)}{2}$

$= 3$

$\therefore a = 3$

Centre of oscillation

$\frac{2+(-4)}{2}$

$= -1$

$\therefore c = -1$

End of Questions in Answer Booklet 1

Question 21 (3 marks)

Solve the following equation for x **3**

$$5^x - \frac{8}{5^x} = 2$$

let $m = 5^x$

$m - \frac{8}{m} = 2$

$m^2 - 8 = 2m$

$m^2 - 2m - 8 = 0$

$(m - 4)(m + 2) = 0$

$\therefore m = 4$ or $m = -2$

$\therefore 5^x = 4$ or $5^x = -2$

$x = \log_5 4$ no solutions

$= \frac{\ln 4}{\ln 5}$

$\therefore x = \frac{\ln 4}{\ln 5}$

$\doteqdot 0.86$

Question 22 (3 marks) **3**

Use the trapezoidal rule with 4 intervals to evaluate $\int_{0.5}^{2.5} \sqrt[3]{\log_e x}\, dx$ correct to 3 decimal places

4 intervals → 5 function values

x	0.5	1.0	1.5	2.0	2.5
y	-0.885	0	0.74	0.885	0.97

$\therefore \int_{0.5}^{2.5} \sqrt[3]{\log_e x} \doteqdot \frac{0.5}{2}\left[(-0.885 + 0.97) + 2(0 + 0.74 + 0.885)\right]$

$\doteqdot 0.834$

Question 23 (5 marks)

(a) Find the x coordinate of the stationary point(s) on the curve $y = 3\log_e x + x^2 - 7x$, where $x > 0$. **2**

$$\frac{dy}{dx} = \frac{3}{x} + 2x - 7$$

Stationary point when $\frac{dy}{dx} = 0$

$$\frac{3}{x} + 2x - 7 = 0$$

$$2x^2 - 7x + 3 = 0$$

$$(2x - 1)(x - 3) = 0$$

$\therefore$ $x = \frac{1}{2}$ or $x = 3$

(b) Hence, determine the nature of each of these stationary point(s) **3**

$$\frac{d^2y}{dx^2} = \frac{-3}{x^2} + 2$$

at $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = -3 \times 4 + 2$

$= -10 < 0$

$\therefore$ maximum turning point at $x = \frac{1}{2}$

at $x = 3$ $\frac{d^2y}{dx^2} = -\frac{3}{9} + 2$

$= \frac{5}{3}$

> 0

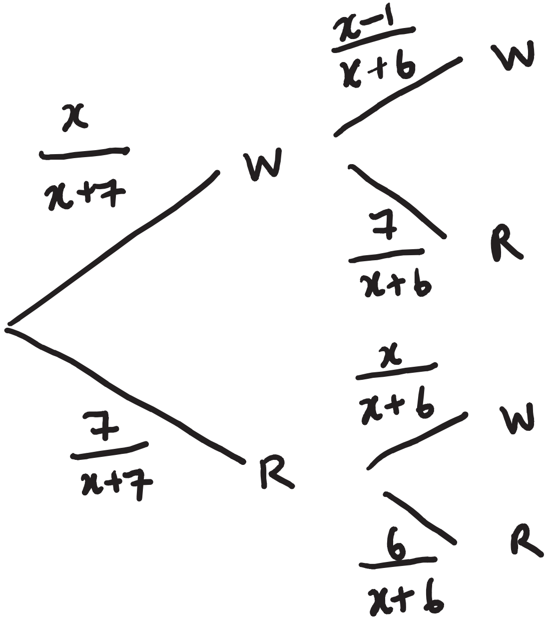
$\therefore$ minimum turning point at $x = 3$

Question 24 (4 marks)

There are some red counters and some white counters in a bag. At the start, 7 of the counters are red and the rest of the counters are white. Woody takes two counters from the bag. First he takes at random a counter from the bag. He does not put the counter back in the bag. Woody then takes at random another counter from the bag.

(a) Let the number of white counters in the bag be x. **2**

Draw a tree diagram that represents the above scenario, showing all relevant information.



There are initially $x+7$ counters in the bag

(b) It is given that the probability that the first counter Woody takes is white, **and** the second counter Woody takes is red is $\frac{21}{80}$. **2**

Find the number of white counters in the bag at the start.

Given $P(WR) = \frac{21}{80}$

$$\frac{x}{x+7} \times \frac{\cancel{7}}{x+6} = \frac{\cancel{21}^{3}}{80}$$

$$80x = 3(x+7)(x+6)$$
$$= 3(x^2+13x+42)$$
$$= 3x^2+39x+126$$
$$3x^2-41x+126=0$$

$$x = \frac{41 \pm \sqrt{169}}{6}$$
$$= \frac{41+13}{6} \text{ or } \frac{41-13}{6}$$
$$= 9 \text{ or } 4\tfrac{2}{3}$$

$\therefore x = 9$ as x is an integer

Question 25 (6 marks)

A geometric sequence is such that its sum to 4 terms is 17 times its sum to 2 terms. It is given that the common ratio of this geometric sequence is positive and not equal to 1.

(a) Find the common ratio of this geometric sequence. **3**

$S_4 = 17 S_2$

$S_4 = \frac{a(r^4-1)}{r-1}$ and $S_2 = a + ar$

$= a(1+r)$

$\therefore \frac{a(r^4-1)}{r-1} = 17a(1+r)$

$r^4 - 1 = 17(r^2-1)$ as $a \neq 0$

$(r^2-1)(r^2+1) = 17(r^2-1)$

$(r^2-1)(r^2+1) - 17(r^2-1) = 0$

$(r^2-1)[r^2+1-17] = 0$

$\therefore (r^2-1)(r^2-16) = 0$

$\therefore r = 4$

as $r \neq 1, -1$ or -4

(b) Given that the 6th term of this geometric sequence is 64. Find the first term. **2**

$T_6 = ar^5$

$\therefore 64 = a(4)^5$

$a = \frac{64}{4^5}$

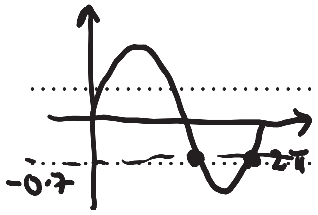
$= \frac{1}{16}$

(c) Does this sequence have a limiting sum? Explain your reasoning. **1**

For a limiting sum to exist, $-1 < r < 1$.
as $r = 4$, this sequence will not have a limiting sum.

Question 26 (3 marks) **3**

Determine the number of solutions for $\sin\theta = -0.7$, where $\theta \in [0, 51\pi]$ and provide reasons for your answer.



$\sin\theta = -0.7$ twice in each period

period $= 2\pi$

$\therefore$ for $\theta \in [0, 50\pi]$ we have 25 periods

$\therefore$ 50 solutions.

For $\theta \in [50\pi, 51\pi]$ no solutions exist as $\sin\theta > 0$ in this interval.

$\therefore$ total of 50 solutions.

Question 27 (3 marks) **3**

The Richter scale defines the magnitude of an earthquake as $M = \log_{10}\left(\frac{I}{S}\right)$ where I is the intensity of the earthquake wave, and S is the intensity of the smallest detectable wave.

An earthquake that registered 6.4 in magnitude was followed by another which was 4 times more intense.

Determine the magnitude of the second earthquake accurate to 1 decimal place.

1st Quake:

$6.4 = \log_{10}\left(\frac{I_1}{S}\right)$

$\frac{I_1}{S} = 10^{6.4}$

2nd Quake

$I_2 = 4I_1$

$\therefore M_2 = \log_{10}\left(\frac{I_2}{S}\right)$

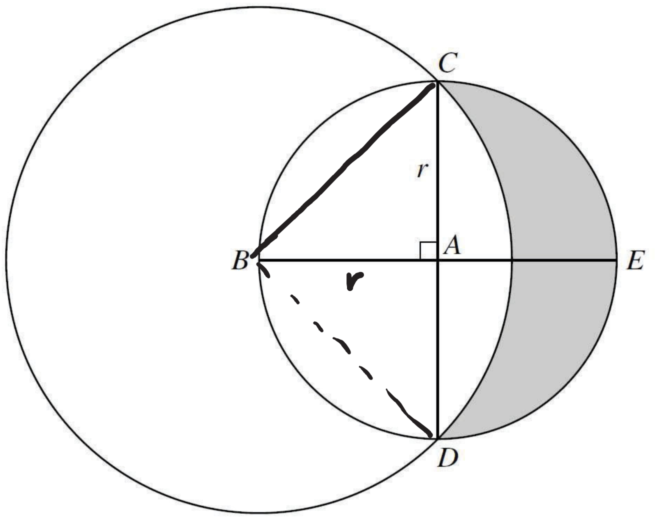
$= \log_{10}\left(\frac{4I_1}{S}\right)$

$= \log_{10}(4 \times 10^{6.4})$

$= \log_{10} 4 + 6.4$

$\doteqdot 7.0$ (to 1 d.p)

Question 28 (3 marks) **3**



The above diagram shows a circle with centre A and radius r. Diameters CAD and BAE are perpendicular to each other. A larger circle has centre B and passes through C and D.

Find the area of the shaded region in terms of r.

using Pythagoras' Theorem in ΔABC, $BC = \sqrt{2}r$ and

$\angle BAC = \frac{\pi}{4}$, hence $\angle BAD = \frac{\pi}{2}$

$A_{\text{segment } CD} = \frac{1}{2}(\sqrt{2}r)^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$= r^2\left(\frac{\pi}{2} - 1\right)$

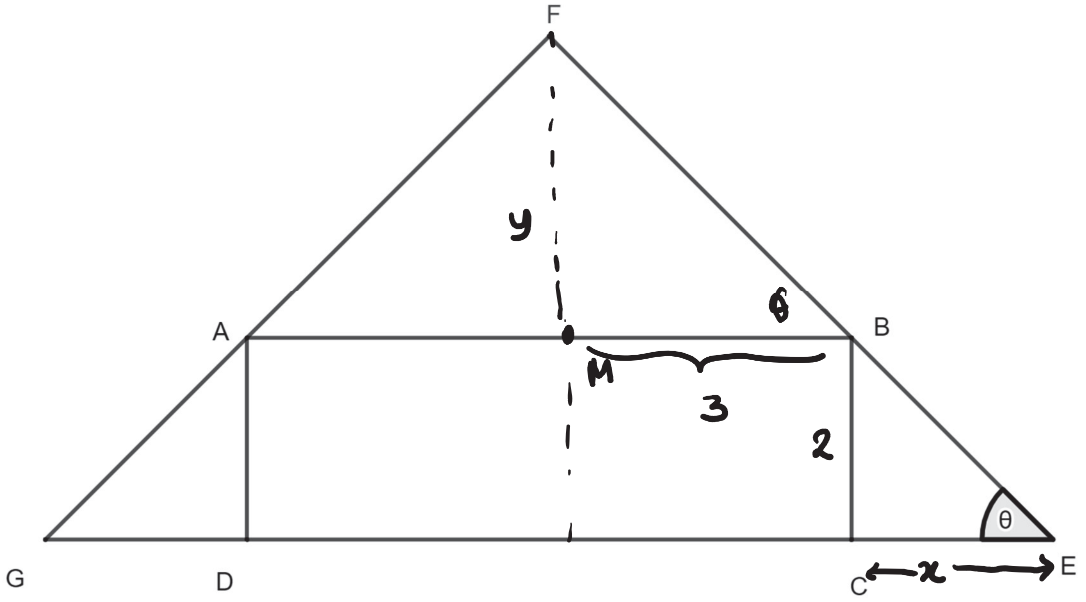
Shaded Area = Area of semicircle − Area of segment

$= \frac{1}{2}\pi r^2 - \left(\frac{\pi r^2}{2} - r^2\right)$

$= \frac{1}{2}\pi r^2 - \frac{\pi r^2}{2} + r^2$

$= r^2$ units^2

Question 29 (7 marks)



A rectangle $ABCD$ of length 6cm and width 2cm is inscribed in an isosceles triangle EFG, where $FG = FE$.

Let $\angle FEG = \theta$

(a) Show that the area of the isosceles triangle can be expressed as **3**

$$A = 12 + 9\tan\theta + 4\cot\theta$$

$A = \frac{1}{2} b \times h$

$b = 2x + 6$ and $h = y + 2$

$\tan\theta = \frac{2}{x}$ $\tan\theta = \frac{y}{3}$

$\therefore x = 2\cot\theta$ $\therefore y = 3\tan\theta$

$\therefore A = \frac{1}{2}(2x+6)(y+2)$

$= (x+3)(y+2)$

$= xy + 2x + 3y + 6$

$= 6 + 2(2\cot\theta) + 3(3\tan\theta) + 6$

$= 12 + 4\cot\theta + 9\tan\theta$

(b) Hence, find the minimum area of the isosceles triangle that inscribes the rectangle. **4**

$$A = 12 + 4(\tan\theta)^{-1} + 9\tan\theta$$

$$\frac{dA}{d\theta} = -4(\tan\theta)^{-2} \cdot \sec^2\theta + 9\sec^2\theta$$

$$= \frac{-4\cancel{\cos^2\theta}}{\sin^2\theta} \times \frac{1}{\cancel{\cos^2\theta}} + \frac{9}{\cos^2\theta}$$

$$= \frac{-4}{\sin^2\theta} + \frac{9}{\cos^2\theta}$$

$$= \frac{-4\cos^2\theta + 9\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

stationary when $\frac{dA}{d\theta} = 0$

$$-4\cos^2\theta + 9\sin^2\theta = 0$$

$$9\sin^2\theta = 4\cos^2\theta$$

$$\tan^2\theta = \frac{4}{9}$$

$\tan\theta = \pm\frac{2}{3}$ as $0 < \theta < 90°$

$$\theta \doteqdot 33°41'$$

Testing for minimum

θ	$30°$	$33°41'$	$45°$
$\frac{dA}{d\theta}$	-4	0	10
	\	—	/

∴ minimum when $\theta \doteqdot 33°41$

∴ $A = 12 + 4 \cdot \frac{3}{2} + 9 \cdot \frac{2}{3}$

$= 24 \text{ cm}^2$

Question 30 (4 marks) **4**

The first four numbers of an arithmetic sequence are $p, 9, 3p - q, 3p + q$.

Find the 2022th term.

$9 - p = (3p+q) - (3p-q)$

$= 2q$

$p + 2q = 9$ ①

Now $9 + 2q = 3p - q$ as $2q$ is common diffaence

$3p - 3q = 9$

$p - q = 3$ ②

① - ②

$3q = 6$

$q = 2$

$\therefore p = 5$

$\therefore$ sequence is 5, 9, 13, 17 ...

$a = 5$, $d = 4$

$\therefore T_{2022} = 5 + (2021) \times 4$

$= 8089$

Question 31 (6 marks)

(a) Given that $y = xe^{-x}$, find $\frac{dy}{dx}$ and hence show that **3**

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$\frac{dy}{dx} = x \cdot -e^{-x} + e^{-x}$

$= -xe^{-x} + e^{-x}$

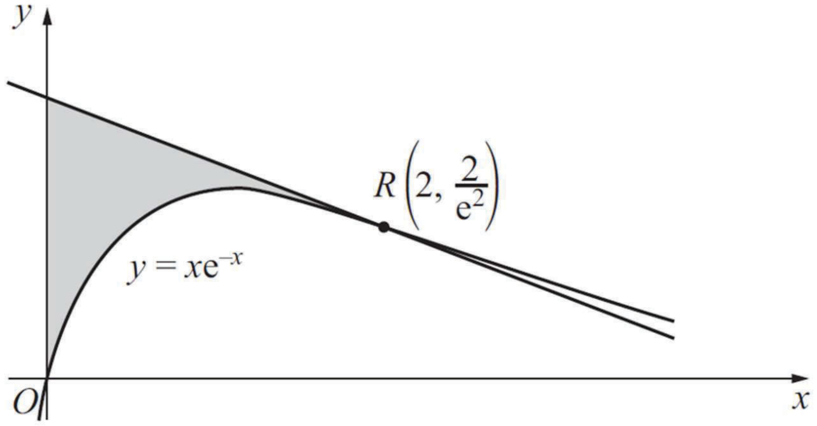
$\therefore \int -xe^{-x} + e^{-x}\,dx = xe^{-x} + c_1$

$-\int xe^{-x}\,dx = xe^{-x} - \int e^{-x}\,dx$

$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + c$

(b) **3**



The diagram above shows part of the curve $y = xe^{-x}$ and the tangent to the curve at the point $R\left(2, \frac{2}{e^2}\right)$.

Find the area of the shaded region bounded by the curve, the tangent and the y axis.

From a) $\frac{dy}{dx} = e^{-x}(1-x)$

when $x = 2$ $m_T = -\frac{1}{e^2}$

$\therefore$ equation of tangent: $y - \frac{2}{e^2} = -\frac{1}{e^2}(x-2)$

$y = -\frac{1}{e^2}x + \frac{2}{e^2}$

$y = -\frac{1}{e^2}x + \frac{4}{e^2}$

Area: $\int_0^2 -\frac{1}{e^2}x + \frac{4}{e^2} - xe^{-x}\,dx$

$= \left[-\frac{1}{2e^2}x^2 + \frac{4}{e^2}x - \left(-xe^{-x} - e^{-x}\right)\right]_0^2$

$= \left[-\frac{1}{2e^2}x^2 + \frac{4}{e^2}x + xe^{-x} + e^{-x}\right]_0^2$

$= \left(-\frac{2}{e^2} + \frac{8}{e^2} + \frac{2}{e^2} + \frac{1}{e^2}\right) - (0 + 0 + 0 + 1)$

$= \frac{9}{e^2} - 1 \quad u^2$

Question 32 (2marks)

Given that $E(aX + b) = aE(X) + b$, where $E(X)$ is the expected value of a discrete random variable X and a and b are constants. **2**

Prove that $Var(aX + b) = a^2\ Var(X)$

$$Var(aX+b) = E((\underset{y}{a}X+b)^2) - [E(aX+b)]^2$$

$$= E(a^2X^2 + 2abX + b^2) - [aE(x)+b]^2$$

$$= a^2E(X^2) + 2a\cancel{b}E(x) + \cancel{b^2} - a^2(E(x))^2 - 2a\cancel{b}E(x) - \cancel{b^2}$$

$$= a^2\left(E(x^2) - (E(x))^2\right)$$

$$= a^2 Var(x)$$

Question 33 (3 marks) **3**

At the point $(2, -5)$ on the curve $y = f(x)$, the tangent has the equation $2x - y - 9 = 0$. Determine the equation of the tangent to the curve $y = 4 - 2f\left(3 + \frac{x}{2}\right)$ at the point $(-2, 14)$, showing all reasoning.

Tangent will be transformed in the same way as $f(x)$

$$y = 4 - 2f\left(3 + \frac{x}{2}\right)$$

$$\Rightarrow -\left(\frac{y-4}{2}\right) = f\left(3 + \frac{x}{2}\right)$$

replace y with $-\left(\frac{y-4}{2}\right)$ and x with $\left(3 + \frac{x}{2}\right)$

$$2\left(3 + \frac{x}{2}\right) - \left(\frac{4-y}{2}\right) - 9 = 0$$

$$12 + 2x + y - 4 - 18 = 0$$

$$y = -2x + 10$$

End of Paper

Section II Extra writing space

If you use this space, clearly indicate which question you are answering.